ON FORMAL AND PERIODIC SOLUTIONS OF THE THREE-BODY PROBLEM

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ABSTRACT:

Highlights on Poincaré's research concerning semiconvergent series are given in this paper. The problem of the vanishing Hessian is analyzed with regard to both linear and non-linear differential equations in a special case studied by that author, showing that the relationships between periodic and formal solutions of the three-body problem should be investigated more closely. In this connection, further investigations should also take into account Barrar's results referred to the problem of normalization of the Hamiltonian in canonical systems.

1. Let us consider the following canonical system of differential equations with two degrees of freedom:

$$(1) \qquad \frac{dx_i}{dt} = \frac{\partial F}{\partial y_i} \qquad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i} \qquad i = 1, 2$$

where F, the Hamiltonian depends on a small parameter μ. It follows from elementary considerations that F admits a development of the form:

$$(2) \qquad F = F_0 + \mu F_1 + \mu^2 F_2 + \dots$$

where F_0 depends only the x set of variables, meanwhile the remaining F_i (i=1, 2, 3 ...) depend on the x set as well as on the y set of variables. (The y's being periodic functions with period 2π).

The question of integrals of system (1) in Celestial Mechanics is closely connected to the vanishing of the Hessian

$$(3) \qquad H(F_0) = \left| \frac{\partial^2 F_0}{\partial x_1 \partial x_2} \right| = 0$$

This is almost a general fact in the three-body problem the restricted three-body problem being an exception. With regard to planetary theory we shall refer later to a result obtained by Moser and Siegel.

The immediate consequence of the vanishing of $H(F_0)$ is that the series which are solutions of system (1) can only satisfy them formally. I have treated these last two problems in several previous papers.

2. Let us now pay attention to the following linear differential equation due to Gyldén:

$$(4) \qquad \frac{d^2x}{dt^2} + q^2x = q_1 x \cos 2t$$

where q^2 and q^1 are some constants. Equation (4) follows from another differential equation due to Gyldén:

$$(5) \qquad \frac{d^2\rho}{d\nu_0^2} + \rho = B$$

where ν_0 is the undisturbed value of the mean longitude: ρ is the difference between the disturbed and undisturbed values of the quantity

$$-\frac{1}{u} = r$$

r being the geocentric distance of the body. (The Moon). B is the lunar disturbed lunar function.

The differential equation (4) is obtained from the non-linear equation (5) by taking an appropriate term from B; after this term has been linearized, the remaining terms of B are neglected. Equation (4) follows, then inmediatly. This equation (4) can be brought to the canonical form (1) by means of elementary transformations of variables. In this particular case the Hamiltonian takes the form:

$$(6) \qquad F = -qx_1 - x_2 + \mu x_1 \sin y_1 \sin y_2$$

We have clearly

$$F_0 = -qx_1 - x_2$$

and then:

$$\left| \frac{\partial^2 F_0}{\partial x_1 \partial x_2} \right| = 0$$

However, our linear differential equation (4) has periodic solutions, the nature of which has extensively by studied by Poincaré and others.

Let us now turn our attention to the non-linear differential equation (Gyldén)

$$(7) \qquad \frac{d^2x}{dt^2} + (q^2 - q_1 \cos 2t)x = \alpha \varphi(x,t)$$

which, clearly, is another form of writing equation (5); ϕ is here a function which can be developed in powers of the variable x, and it is also expressed in terms of the following arguments:

$$\lambda_2 t, \lambda_3 t, \ldots\ldots \lambda_n t$$

where: $\lambda_2 = 2$

When we write:

$$y = \frac{dx}{dt} \qquad y_i = \lambda_i t$$

$$\varphi(x,t) = \frac{d\Psi}{dx}$$

where ψ has the same properties as ϕ. Let us now assume that we have replaced $\lambda_i t$ by y in $q_1 \cos 2t$, ϕ and ψ. By introducing n-1 auxiliary variables $x_2, x_3, \ldots\ldots x_n$, and putting:

$$(8) \qquad F = \frac{y^2}{2} - \alpha\Psi + \frac{x^2}{2}\left(q^2 - q_1 \cos y_2\right) - \sum \lambda_i x_i$$

we obtain the following canonical system:

$$(9) \qquad \frac{dx}{dt} = \frac{\partial F}{\partial y} \qquad \frac{dy}{dt} = -\frac{\partial F}{\partial x}$$

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i} \qquad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i} \qquad (i = 2, 3 \ldots n)$$

By putting:

$$x = \frac{1}{q}\sqrt{2x_1}\cos y_1$$

$$y = q\sqrt{2x_1}\sin y_1$$

we have that:

$$x\,dy \sim x_1\,dy_1$$

is an exact differential, and then, F will be periodic with respect to the y's. We shall have:

$$qx^2 + y^2 = 2qx_1$$

α is a small parameter. When $\alpha = 0$, we get:

$$F_0 = qx_1 - \frac{q_1}{q_2}\, x \cos^2 y_1 \cos y_2 - \sum \lambda_i x_i$$

F_0 depends here on the x set as well as on the y set. The last step is to seek an undisturbed Hamiltonian F_0 where the y's be absent. By taking now the y's such that:

$$y_i' = \lambda_i t + \varpi_i$$

where the λ_i and $\bar{\omega}$: are constants; we shall also have

$$y_1' = -ht + \varpi \qquad \text{and when:}$$

$$x_i = x_i' \qquad y_i = y_i' \qquad i > 2.$$

we can construct a new Hamiltonian F such that for $\alpha = 0$ we get:

$$(10) \qquad F_0 = h x_1' - \lambda^2 x_2' - \lambda_2 x_2' - \dots - \lambda_n x_n'$$

corresponding to the set of differential equations

$$(11) \qquad \frac{dx_i'}{dt} = \frac{\partial F}{\partial y'}, \qquad \frac{dy_i'}{dt} = -\frac{\partial F}{\partial x_i'} \quad (i = 1, 2, \dots n)$$

Clearly:

$$\left| \frac{\partial^2 F_0}{\partial x_i \partial x_j} \right| = 0$$

and consequently equation (6) admits only formal solutions. It is then evident that the systems (1), with two degrees of freedom, and respectively, system (11) with n degrees of freedom have resp. periodic solutions and formal solutions, despite that in both cases the Hessian $H(F_0)$ - vanishes. It should be remembered here that equation (4) is a particular case of equation (6), when α is put equal to zero.

We should now mention here an important result obtained by Poincaré in his search of differential equations of Celestial Mechanics. He has shown that equations of the form:

$$\frac{d^2x}{dt^2} - \alpha x = \mu \varphi(x, t, \mu)$$

where α is a positive constant, μ a small parameter, and ϕ is a power series in the parameter μ, the coefficients of which depend on trigonometrical terms with $\lambda_i t$ as arguments, will admit convergent solutions in the following two cases:

1) When ϕ depends on only one argument λt, convergences takes place when α is a positive (or negative) number.

2) When ϕ depends on several (finite number) of arguments $\lambda_i t$, convergence takes place only when α is positive.

We shall point put finally in this paragraph that Siegel and Moser has succeeded to construct planetary theories with a non-vanishing Hessian Fo. The series of Celestial Mechanics do converge in this case then.

3. There is still another way of approaching the problem of convergence in Celestial Mechanics. In fact, given the Hamiltonian:

$$(12) \qquad H = \lambda_1 p_1 + \lambda_2 p_2 + \sum_{i+j=2}^{\infty} A_{ij}(q_1, q_2)\, p_1^i p_2^j$$

where the constant term and the linear periodic terms in q_1 and q_2 are absent. If λ_1/λ_2 is irrational, for any preassigned integer n the Hamiltonian (12) can be reduced to the form:

$$H = \lambda_1 p_1 + \lambda_2 p_2 + \sum_{i+j=2}^{n-1} A_{ij} P_1^i P_2^j + \sum_{i+j=n}^{\infty} A_{ij}(Q_1, Q_2) \cdot P_1^i P_2^j$$

It is then possible to eliminate periodic terms up to an arbitrarily high order, although the circle of convergence diminishes in each step. Assuming now that by a cononical (convergent) transformation we can write a new Hamiltonian

$$H(p_i q_i) = \lambda_1 P_1 + \lambda_2 P_2 + A P_1^2 + B P_1 P_2 + C P_2^2 + + \sum_{i+j=3}^{\infty} A_{ij} P_1^i P_2 = H(P)$$

is can be asserted (Barrar) that for a given B_{ij} (q_1, q_2) and given $\varepsilon>0$, there are arbitrarily small C_{ij} (q_1, q_2) such that the transformation to normal form of

$$A_{ij} + \varepsilon B_{ij}(q_1, q_2) + C_{ij}(q_1, q_2)$$

will not converge in a whole neighborhood of the origin.

4. As a last point I would like turn my attention to a remarkable differential equation which appears in planetary theory. The equation reads:

$$\frac{d^2 x}{dt^2} + P(t)\, x = 0 \tag{13}$$

where $x = E - E_0$ is the difference between the disturbed and undisturbed values of the eccentric anomaly. $P(t)$ is a trigonometric series in multiples of the argument E, the coefficients being powers of the eccentricity. $P(t)$ also contains one or several terms with the disturbing mass as a factor. The process of obtaining this differential equation has been described in a previous paper (Altavista). Equation (13) follows from the more general one:

$$\frac{d^2 x}{dt^2} + P_0(t)\, x = \mu\, \varphi(x, t) \tag{14}$$

where ϕ is the disturbing function, μ is the disturbing mass, and $P_0(t)$ is a trigonometrical series in multiples of the argument E; the coefficients are powers of the eccentricity. This equation (14) can be obtained by applying to the general equations of the three-body problem in rectangular coordinates, the method of the variation of the parameters in the second version of it, as devised by Lagrange himself. Then a selected term is taken from the right-hand side of equation (14) and then linearized. This new linear term can then be combined with the linear term of the left-hand side of equation (14) and equation (13) follows at once when the remaining terms from the right-hand side of equation (14) are neglected. The same process can, of course be applied to several terms of the second member of (14).

Equation (13) is now a linear differential equation with a periodic coefficient, and its solutions may be stable or unstable according to the particular values of the parameters. The convergence (or divergence) of solutions takes place for some finite and continuous ranges of the parameters. It is naturally interesting to find out the quality of the solutions for the remaining orbital elements a, e, I, ω, Ω, and ε respectively. With this object we must pay attention to the set of linear variational equations set up by Lagrange for solving this problem. These equations read:

$$(15)\qquad \begin{aligned} &\sum_{i=1}^{3} x_i^{(j)}\delta x_i = 0 \\ &\sum_{i=1}^{3} X_i^{(j)}\delta x_i = Y^{(j)}(E)\delta E \end{aligned}$$

where the δx_i are respectively the variations of the directional cosines P_x, P_y, P_z, Q_x, Q_y, Q_z, the semi-major axis a, the semimenor axis b, and the eccentricity e. The coefficients in equations (15) depend on the undisturbed values of the keplerian elements and $Y^{(j)}(E)$ contains the disturbed value of the eccentric anomaly as well. We then have six equations with eight unknowns. The set (15) can be simplified by using the well-known relationships

$$(16)\qquad \begin{aligned} &P_x^2 + P_y^2 + P_z^2 = 1 \\ &Q_x^2 + Q_y^2 + Q_z^2 = 1 \\ &P_x Q_x + P_y Q_y + P_z Q_z = 0 \end{aligned}$$

By applying appropriate factors to system (15) and taking into account relationships (16) one can get the new system:

$$(17)\qquad \begin{aligned} &\sum_{i=1}^{2} x_i^{*(j)}\delta x_i^{*} = 0 \\ &\sum_{i=1}^{2} X_i^{*(j)}\delta x_i^{*} = Y^{*(j)}(E)\delta E \end{aligned}$$

Here the new unknowns are the semi-major axis a, the eccentricity e and, for instance, the variations of the directional cosines Q_x, Q_y, Q_z. As we have here only four equations with five unknowns, we can complete it by adding the equation:

$$(18)\qquad V_E^2 = k^2\left(\frac{2}{r_E} - \frac{1}{a_o}\right)$$

where

$$V_E'^2 = \dot{x}_E^2 + \dot{y}_E^2 + \dot{z}_E^2$$

with:

$$\dot{x}_E = -a_o P x_o \sin E\,\dot{E} - b_o Q x_o \cos E\,\dot{E}, \quad (x = x, y, z)$$

the index $_o$ refers to the undisturbed values of the elements. We have also:

$$(19)\qquad r_E = a_o\,(1 - e_o \cos E)$$

$$(20) \quad \frac{dE}{dt} = \frac{1}{r_E a_0^{1/2}}$$

According to the principles settled by Lagrange in his second version of his method of the variation of the parameters we must apply to (18) the linear operator δ, keeping fixed the time t, so as to obtain the fifth variational equation which must be added to system (17) for solving the problem. It is now clear that if the solution δE given by equation (13) is convergent, the variational equation will provide convergent solutions for the remaining set of keplerian elements. It follows, from the above discussions, that the vanishing of the Hessian is not an essential point to set up the convergence of the solutions of the planetary set of differential equations.

We should also remark an important difference between both processes for obtaining equations (4) and (13) respectively:

$$(4) \quad \frac{d^2x}{dt^2} + (q_1^2 - q \cos 2t)\, x = 0$$

$$(13) \quad \frac{d^2x}{dt^2} + P(t)\, x = 0$$

In fact, equation (4) follows from equation (5) whose linear term has a (secular) factor 1 multiplying the linear term. On the other hand the coefficient P(t) in equation (13) appears after combining $P_0(t)$ from (14) plus a term from the second member of this same equation. In other words, the coefficient of the linear term in equation (13) has a trigonometrical structure from his very origin.

It is clear from the above discussions, that the solution given by equation (13) improve the results provided by the oldest methods. In this sense, the merit of Lagrange's method of the variation of the parameters in his second version must be emphasized.

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